

CORRELATIONS IN ENTANGLED STATES

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Entangled EPR spin pairs can be treated using the statistical ensemble interpretation of quantum mechanics. As such the singlet state results from an ensemble of spin pairs each with its own specific axis of quantization. This axis acts like a quantum mechanical hidden variable. If the spins lose coherence they disentangle into a mixed state that contains classical correlations. In this paper an infinitesimal phase decoherence is introduced to the singlet state in order to reveal more clearly some of the correlations. It is shown that a singlet state has no classical correlations.

Keywords: Statistical ensemble; entanglement; disentanglement; quantum correlations; EPR states; Bell's inequalities; quantum non-locality.

1. Introduction

The fundamental questions of quantum mechanics are rooted in the philosophical interpretation of the wave function.¹ At the time these were first debated, covering the fifty or so years following the formulation of quantum theory, the arguments were based primarily on gedanken experiments.² Today the situation has changed with numerous experiments now possible that can guide us in our search for the true nature of the microscopic world, and how *The Infamous Boundary*³ to the macroscopic world is breached. The current view is based upon pivotal experiments, first performed by Aspect⁴ showing that quantum mechanics is correct and Bell's inequalities⁵ are violated. From this the non-local nature of quantum mechanics appears firmly entrenched in physics leading to new technologies, notably those where non-locally is considered to be fundamental such as for⁶ quantum "teleportation".

The singlet state, formed from two spins of 1/2 magnitude is the simplest wave function that displays entanglement. Entanglement is considered necessary for the non-local properties of the wave function. Experimental studies of a system in a singlet state involve determining the correlations that can exist between the two spins even after they have separated and are measured by instruments that detect one spin in the pair but not the other. Two-spin correlation is defined by a function

$$E(\mathbf{a}, \mathbf{b}) = \mathbf{a} \cdot \langle \boldsymbol{\sigma}^1 \boldsymbol{\sigma}^2 \rangle \cdot \mathbf{b} - \mathbf{a} \cdot \langle \boldsymbol{\sigma}^1 \rangle \langle \boldsymbol{\sigma}^2 \rangle \cdot \mathbf{b} \quad (1)$$

Here the Pauli spin vectors are σ^i for each spin and the vectors \mathbf{a} and \mathbf{b} determine the direction of the detectors (polarizer angles or magnetic field directions). The purpose of this paper is to examine some of the correlations that can exist in the correlation function, $E(\mathbf{a}, \mathbf{b})$. More specifically the correlations are determined by the spin-pair density operator, ρ^{12}

$$\mathbf{a} \cdot \langle \sigma^1 \sigma^2 \rangle \cdot \mathbf{b} = \mathbf{a} \cdot \text{Tr}_{12} [\rho^{12} \sigma^1 \sigma^2] \cdot \mathbf{b} \quad (2)$$

If a pure singlet state is measured, then it is well known that the correlation is given by

$$E_E(\mathbf{a}, \mathbf{b}) = -\mathbf{a} \cdot \mathbf{b} = -\cos \theta_{ab} \quad (3)$$

where the subscript 'E' is introduced to indicate a fully entangled state.

Although experimental production of entangled states is now considered routine, the states might contain small mismatches in the phases between the two spins. In order to study this, we first define the singlet state in terms of an arbitrary axis of quantization, after which we show that a small phase difference between the two spins leads to the existence of different types of correlations that display different properties.

2. The Statistical Singlet State

The singlet state of a pair of spins as introduced by Bohm and Aharonov⁷ is used here. In that case the two states of a spin 1/2 are given by

$$|+\rangle_{\theta_i \phi_i}^i = \begin{pmatrix} \cos(\frac{\theta_i}{2}) \\ \sin(\frac{\theta_i}{2})e^{i\phi_i} \end{pmatrix} \quad \text{and} \quad |-\rangle_{\theta_i \phi_i}^i = \begin{pmatrix} -\sin(\frac{\theta_i}{2})e^{-i\phi_i} \\ \cos(\frac{\theta_i}{2}) \end{pmatrix} \quad (4)$$

where $i=1$ or 2 . The angles θ_i and ϕ_i define an axis along which the spin is quantized which we denote by the unit vector $\hat{\mathbf{P}}_i$ and this is relative to a basis defined with respect to the z axis, $|\pm\rangle_{00}^i$ with $\theta_i = 0, \phi_i = 0$. These spin states are eigenstates of the Pauli spin operator, $\sigma_{\theta_i \phi_i}^i$ with eigenvalues,

$$\sigma_{\theta_i \phi_i}^i |\pm\rangle_{\theta_i \phi_i}^i = \pm |\pm\rangle_{\theta_i \phi_i}^i \quad (5)$$

Consider now a state formed from the two spin as follows,

$$|\Psi_{12}^{\theta_1 \phi_1, \theta_2 \phi_2 -}\rangle = \frac{1}{\sqrt{2}} \left(|+\rangle_{\theta_1 \phi_1}^1 |-\rangle_{\theta_2 \phi_2}^2 - |-\rangle_{\theta_1 \phi_1}^1 |+\rangle_{\theta_2 \phi_2}^2 \right) \quad (6)$$

Although this state is entangled, it is not a singlet state since the angles are unequal, $\theta_1 \neq \theta_2$ and $\phi_1 \neq \phi_2$. In the following, it is assumed, for simplicity, that the polar

angles are equal $\theta = \theta_1 = \theta_2$. Inserting Eq. (4) into Eq. (6) gives, therefore

$$|\Psi_{12}^{\theta\phi_1,\theta\phi_2-}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos \frac{\theta}{2} \sin \frac{\theta}{2} (e^{-i\phi_1} - e^{-i\phi_2}) \\ \cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2} (e^{-i(\phi_1-\phi_2)}) \\ -\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} (e^{+i(\phi_1-\phi_2)}) \\ \cos \frac{\theta}{2} \sin \frac{\theta}{2} (e^{i\phi_1} - e^{i\phi_2}) \end{pmatrix}. \tag{7}$$

This state is anisotropic and depends upon the axes of quantization of the two spins. However, no matter what axes of quantization exists, if the two exactly coincide (*i.e.* $\theta_1 = \theta_2$ and $\phi_1 = \phi_2$) for a given spin pair, the singlet state results,

$$|\Psi_{12}^{\theta\phi,\theta\phi-}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \quad \text{for all } \theta \text{ and } \phi \tag{8}$$

This state is, of course, completely isotropic and arises for all possible values of θ and ϕ . In other words, for any axis of quantization, the singlet state in Eq. (8) is always isotropic. Expressing this well known result in terms of state vectors gives

$$|\Psi_{12}^-\rangle = \frac{1}{\sqrt{2}} \left[|+\rangle_{\hat{\mathbf{P}}} |-\rangle_{\hat{\mathbf{P}}} - |-\rangle_{\hat{\mathbf{P}}} |+\rangle_{\hat{\mathbf{P}}} \right]. \tag{9}$$

It is immediately evident that the RHS depends upon the choice of axis of quantization, but the LHS does not. To this extent, one can envision the axis of quantization as a hidden variable. The hidden variable theory introduced by Bell⁵ for a spin 1/2, makes use of the axis of quantization. It is this axis, in the case of the singlet state, that provides the ‘element of reality’ for understanding experiments involving separated entangled particles.⁸ The statistical view of the EPR singlet state is that of an ensemble of spin pairs with each pair having its own unique $\hat{\mathbf{P}}$. If the EPR source is isotropic, then an infinite number of quantization axes are possible; each equally probable over the surface of a sphere; all of which result in isotropic singlet states; and all of which are indistinguishable one from the other.

Obviously a small change in the axis of quantization for a given EPR pair in the ensemble, (for example if they lose phase coherence so that $\phi_1 \neq \phi_2$), will cause the singlet state to lose its isotropy. The ensemble then becomes a mixed state with the different axes of quantization being revealed. If, however, the EPR pairs only lose azimuthal coherence, $\phi_1 \neq \phi_2$, they still retain correlation by virtue of the fact that the polar angles are equal, $\theta_1 = \theta_2$. If, finally, $\theta_1 \neq \theta_2$, then all spin correlation between the initial EPR pair is lost and one has an ensemble of spins, each characterized by its own unique axis of quantization, $\hat{\mathbf{P}}$.

The use of an ensemble here is not a construction for convenience but rather depicts a physical system whereby the singlet state can be formed from any EPR pair with arbitrary angles, θ and ϕ .

This treatment serves to show not only that the entangled singlet state can be characterized by an ensemble of spins with different quantization axes but also displays the sensitivity of the Bell states to changes in the angles that define entanglement.

3. Classical Correlations

In this Section it is shown that correlation exists between separated EPR pairs even when the singlet state disentangles⁹ into a product state devoid of entanglement. The functional form of the correlation is identical for both entangled and disentangled spin pairs, being $\cos \theta_{ab}$, and only differs by a constant factor. In other words, long range correlation is predicted from disentangled EPR pairs even though these correlations obey Bell’s inequalities and therefore obey Einstein locality.

As discussed above, the correlation between the spins of a singlet state depends only on the angle between the two detectors vectors,

$$E_E(\mathbf{a}, \mathbf{b}) = -\mathbf{a} \cdot \mathbf{b} = -\cos \theta_{ab} \tag{10}$$

The density operator for the fully entangled state is

$$\rho_E^{12} = |\Psi_{12}^- \rangle \langle \Psi_{12}^-| \tag{11}$$

It is possible to write $\cos \theta_{ab}$ in terms of some coordinate frame of reference. Choosing this to be the axis of quantization, $\hat{\mathbf{P}}$, then for a particular spin pair,

$$E_E(\mathbf{a}, \mathbf{b}) = -\cos \theta_{ab} = -\cos \theta_a \cos \theta_b - \sin \theta_a \sin \theta_b \cos(\phi_b - \phi_a) \tag{12}$$

The angles on the RHS are defined for this EPR pair as

$$\cos \theta_a = \mathbf{a} \cdot \hat{\mathbf{P}} \quad \text{and} \quad \cos \theta_b = \mathbf{b} \cdot \hat{\mathbf{P}} \tag{13}$$

If there is complete loss of correlation due to second term on the RHS of Eq. (12), then the state loses its entanglement property and is disentangled,⁸ subscript ‘D’,

$$E_D(\mathbf{a}, \mathbf{b}) = -\cos \theta_a \cos \theta_b = \mathbf{a} \cdot \hat{\mathbf{P}} \hat{\mathbf{P}} \cdot \mathbf{b} \tag{14}$$

As such, the disentangled state is anisotropic and depends upon the axis of quantization of that particular EPR pair. Whereas ensemble averaging is unnecessary for the isotropic entangled state, it is necessary for the disentangled state when repeated measurements are made on disentangled EPR pairs. If every axis of quantization is equally probable, then the ensemble averaged correlation, denoted by a bar, is given by

$$\overline{E_D(\mathbf{a}, \mathbf{b})} = \overline{-\mathbf{a} \cdot \hat{\mathbf{P}} \hat{\mathbf{P}} \cdot \mathbf{b}} = -\frac{1}{3} \cos \theta_{ab} \tag{15}$$

On the other hand, if the EPR pair consists of photons that move in the z direction, the ensemble average is over the xy plane only. For spins of $1/2$ magnitude, this leads to,

$$\overline{E_D(\mathbf{a}, \mathbf{b})} = \overline{-\mathbf{a} \cdot \hat{\mathbf{P}} \hat{\mathbf{P}} \cdot \mathbf{b}} = -\frac{1}{2} \cos \theta_{ab} \tag{16}$$

In either case, the disentangled correlations differ from the correlation for a fully entangled state by only a numerical factor. Moreover, it can be shown that the disentangled correlations always obey Bell's inequalities whereas, of course, the entangled correlations can violate Bell's inequalities. We can therefore conclude that the correlations that obey Bell's inequalities are classical and those that lead to violation of Bell's inequalities are quantum,

$$E_E(\mathbf{a}, \mathbf{b}) = - \overbrace{\cos \theta_a \cos \theta_b}^{\text{Classical}} - \overbrace{\sin \theta_a \sin \theta_b \cos(\phi_b - \phi_a)}^{\text{Quantum}} \quad (17)$$

Following from Eq. (16), ensemble averaging gives equal contribution from each,

$$\overline{E_D(\mathbf{a}, \mathbf{b})} = - \overbrace{\frac{1}{2} \cos \theta_{ab}}^{\text{Classic}} - \overbrace{\frac{1}{2} \cos \theta_{ab}}^{\text{Quantum}} \quad (18)$$

This treatment suggests that the entangled singlet state contains both classical and quantum correlations in equal amounts when EPR photons are prepared for Aspect-type experiments. However, the situation is not so straightforward. As shown in the next Section, a fully entangled state contains no classical correlations whatsoever. Rather the contribution from disentanglement is completely canceled by a complementary term that arises from the quantum contribution.

Since the correlations due to entanglement and disentanglement differ only by a numerical factor, caution should be used when a sinusoidal response is observed in two photon coincidence experiments. Such a response is insufficient to ensure that the spin pairs are entangled. In addition to a sinusoidal correlation, the data must also be shown to violate Bell's inequalities.

4. Quantum Correlations

The loss of quantum correlations due to disentanglement, which is discussed in Section 3, can be realized by randomizing the phases of the two spins that were initially entangled. This leaves only the classical correlations.

In this Section, we go to the other limit and assume an infinitesimal phase mismatch, δ , between the EPR pair. The resulting correlations are shown to be purely quantum in origin. Specifically, we allow the phases in Eq. (7) to differ as follows,

$$\phi^+ = \phi + \frac{1}{2} \delta \quad \text{for spin 1 and} \quad \phi^- = \phi - \frac{1}{2} \delta \quad \text{for spin 2} \quad (19)$$

This definition reduces the entanglement in the resulting state and causes it to be anisotropic. As such, the axes of quantization of the two spins are revealed and they are, by virtue of the phase mismatch, different.

$$|\Psi_{12}^-(\mathbf{P}^+, \mathbf{P}^-)\rangle = \frac{1}{\sqrt{2}} \left(|+\rangle_{\mathbf{P}^+} |-\rangle_{\mathbf{P}^-}^2 - |-\rangle_{\mathbf{P}^+} |+\rangle_{\mathbf{P}^-}^2 \right) \quad (20)$$

The orientation of the axes of quantization are defined by $\hat{\mathbf{P}}^\pm = (\theta, \phi \pm \delta/2)$ so the corresponding density operator is

$$\rho^{12}(\mathbf{P}^+, \mathbf{P}^-) = |\Psi_{12}^-(\mathbf{P}^+, \mathbf{P}^-)\rangle \langle \Psi_{12}^-(\mathbf{P}^+, \mathbf{P}^-)| \quad (21)$$

The expectation value is

$$\begin{aligned} \mathbf{a} \cdot \langle \sigma^1 \sigma^2 \rangle \cdot \mathbf{b} &= \text{Tr}_{12} \left[(\sigma_a^1 \sigma_b^2) \left(|\Psi_{12}^-(\mathbf{P}^+, \mathbf{P}^-)\rangle \langle \Psi_{12}^-(\mathbf{P}^+, \mathbf{P}^-)| \right) \right] \\ &= \frac{1}{2} \left(\underbrace{\left[\frac{1}{\mathbf{P}^+} \langle +|\sigma_a^1|+ \rangle_{\mathbf{P}^+}^1 \frac{1}{\mathbf{P}^-} \langle -|\sigma_b^2|- \rangle_{\mathbf{P}^-}^2 + \frac{1}{\mathbf{P}^+} \langle -|\sigma_a^1|- \rangle_{\mathbf{P}^+}^1 \frac{1}{\mathbf{P}^-} \langle +|\sigma_b^2|+ \rangle_{\mathbf{P}^-}^2 \right]}_{\text{Classic}} \right. \\ &\quad \left. - \underbrace{\left[\frac{1}{\mathbf{P}^+} \langle -|\sigma_a^1|+ \rangle_{\mathbf{P}^+}^1 \frac{1}{\mathbf{P}^-} \langle +|\sigma_b^2|- \rangle_{\mathbf{P}^-}^2 - \frac{1}{\mathbf{P}^+} \langle +|\sigma_a^1|- \rangle_{\mathbf{P}^+}^1 \frac{1}{\mathbf{P}^-} \langle -|\sigma_b^2|+ \rangle_{\mathbf{P}^-}^2 \right]}_{\text{Quantum}} \right) \end{aligned} \quad (22)$$

where $\sigma_a^1 = \mathbf{a} \cdot \sigma^1$ and $\sigma_b^2 = \mathbf{b} \cdot \sigma^2$. Since the system now contains anisotropy, with each spin pair displaying different axes of quantization, the result must be eventually ensemble averaged after the matrix elements have been calculated. The ensemble averaging is not undertaken here so that Eq. (22) describes the correlations between a specific entangled spin pair that displays this phase mismatch.

Evaluation of the matrix elements is straightforward. The first two terms in Eq. (22) are diagonal and correspond to the classical terms of Section 3. The result is

$$\begin{aligned} &\frac{1}{2} \left(\left[\frac{1}{\mathbf{P}^+} \langle +|\sigma_a^1|+ \rangle_{\mathbf{P}^+}^1 \frac{1}{\mathbf{P}^-} \langle -|\sigma_b^2|- \rangle_{\mathbf{P}^-}^2 + \frac{1}{\mathbf{P}^+} \langle -|\sigma_a^1|- \rangle_{\mathbf{P}^+}^1 \frac{1}{\mathbf{P}^-} \langle +|\sigma_b^2|+ \rangle_{\mathbf{P}^-}^2 \right] \right) \\ &= -\mathbf{a} \cdot \hat{\mathbf{P}}^+ \hat{\mathbf{P}}^- \cdot \mathbf{b} \end{aligned} \quad (23)$$

These differ from the result in Section 3 only by the phase mismatch of the quantization axes and, of course, this difference disappears in the limit that δ goes to zero.

Evaluation of the last two, quantum, terms of Eq. (22) gives,

$$\begin{aligned} &-\frac{1}{2} \left(\left[\frac{1}{\mathbf{P}^+} \langle -|\sigma_a^1|+ \rangle_{\mathbf{P}^+}^1 \frac{1}{\mathbf{P}^-} \langle +|\sigma_b^2|- \rangle_{\mathbf{P}^-}^2 + \frac{1}{\mathbf{P}^+} \langle +|\sigma_a^1|- \rangle_{\mathbf{P}^+}^1 \frac{1}{\mathbf{P}^-} \langle -|\sigma_b^2|+ \rangle_{\mathbf{P}^-}^2 \right] \right) \\ &= [-\mathbf{a} \cdot \mathbf{P}^+ \mathbf{P}^- \cdot \mathbf{b} - a_z b_z - [a_x b_x + a_y b_y] \cos \delta + [a_x b_y - a_y b_x] \sin \delta] \end{aligned} \quad (24)$$

This result demonstrates that the quantum terms, *i.e.* coming from the off-diagonal matrix elements in Eq. (22), contain a term, $\mathbf{a} \cdot \mathbf{P}^+ \mathbf{P}^- \cdot \mathbf{b}$, that exactly cancels the classical contribution of Eq. (23). Putting the classical and quantum contributions together gives,

$$\begin{aligned} \mathbf{a} \cdot \langle \sigma^1 \sigma^2 \rangle \cdot \mathbf{b} &= \\ &= \left[\underbrace{\mathbf{a} \cdot \mathbf{P}^+ \mathbf{P}^- \cdot \mathbf{b}}_{\text{Classic}} - \underbrace{\mathbf{a} \cdot \mathbf{P}^+ \mathbf{P}^- \cdot \mathbf{b} + a_z b_z + [a_x b_x + a_y b_y] \cos \delta - [a_x b_y - a_y b_x] \sin \delta}_{\text{Quantum}} \right] \end{aligned} \quad (25)$$

The only correlations present in a fully entangled singlet state are quantum in origin. Clearly for such a state the classical correlation cancels, and after the phase

mismatch is taken to its zero limit, one retrieves the usual quantum mechanical result that contains no classical contributions,

$$\begin{aligned} \mathbf{a} \cdot \langle \boldsymbol{\sigma}^1 \boldsymbol{\sigma}^2 \rangle \cdot \mathbf{b} = \\ -a_z b_z - [a_x b_x + a_y b_y] \cos \delta + [a_x b_y - a_y b_x] \sin \delta \xrightarrow{\delta \rightarrow 0} = -\cos \theta_{ab} \end{aligned} \quad (26)$$

On the other hand, if the quantum terms are removed by disentanglement, only the classical term survives which gives Eq. (14) (before ensemble average) and, depending on the distribution of axes of quantization, results similar to Eqs. (15) and (16) after ensemble averaging.

Finally, the correlated state is independent of the orientation of the detection devices. That is the correlations in an EPR pair do not depend upon the detectors so that the setting of the detector at one location to measure on spin, has no effect on its distant partner. Rather the correlations are a property of the system alone, and can be expressed as components of a second rank tensor,

$$\begin{aligned} \langle \boldsymbol{\sigma}^1 \boldsymbol{\sigma}^2 \rangle = -\hat{z}\hat{z} - [\hat{x}\hat{x} + \hat{y}\hat{y}] \cos \delta + [\hat{x}\hat{y} - \hat{y}\hat{x}] \sin \delta \\ = -\hat{z}\hat{z} - [\underline{\mathbf{U}} - \hat{z}\hat{z}] \cos \delta - \hat{z} \cdot \underline{\underline{\boldsymbol{\varepsilon}}} \sin \delta \xrightarrow{\delta \rightarrow 0} = -\underline{\mathbf{U}} \end{aligned} \quad (27)$$

where $\underline{\mathbf{U}}$ is the completely symmetric second rank identity tensor with components, $(\underline{\mathbf{U}})_{ij} = \delta_{ij}$, where δ_{ij} is the Kronecker delta function, and $\underline{\underline{\boldsymbol{\varepsilon}}}$ is the completely antisymmetric third rank Levi Civita tensor.

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