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Dedicated in Memoriam to Brian G. Wybourne, lately of: Fizika Instytut., N. Kopernicus Uniwersytet, Toruń Poland.

Branching Rules & Determinacy Criteria for $SU(m) \times \mathcal{S}_{2n} \downarrow \mathcal{G}$ Automorphic NMR Spin Symmetries: Roles of Schur (λ) Decomposition & Multicolour Lattice-point Projective Modelling (of $\{\chi_i\}(C_i(A_5))$ Invariances) in $[\lambda](SU(3) \times \mathcal{S}_{20} \downarrow A_5)$ Mapping.

B.C. Santuary,

Dept. of Chemistry, McGill University,

801 Sherbrooke W, Montreal P. Que., Canada H3A 2K6; {Bryan.Sanctuary@mcgill.ca}

F.P. Temme,

Dept. of Chemistry, Queen's University, Kingston-ON,

Canada, K7L 3N6; {E-mail: Temmef@tantalus.chem.queensu.ca}. .

Abstract. The role of both (λ) Schur decompositions and $\{\chi_i\}(C_i)(\mathcal{S}_{2n} \downarrow \mathcal{G})$ multicolour lattice-point projective modelling of polyhedral combinatorics is examined in establishing the branching rules for $\mathcal{S}_{2n} \supset \mathcal{G}$ automorphic spin symmetries (on spin-alone abstract space) typical of identical 2n-fold higher $I_i \geq 1$ NMR spin ensembles, and their universal (i.e., to-(λ_{SA})) determinacy. The use of algorithmic polyhedral combinatorics(PC) of $\mathcal{S}_{2n}$-invariants of uniform spin (sub)systems governed by democratic recoupling, and recent cycle-index studies of multinomial combinatorial group irreps, provides much context here. Because of the $|\mathcal{S}_{2n}|$ order encountered in $SU(3) \times \mathcal{S}_{20}$ subduction, a hierarchical algorithmic approach (i.e., typical of inverse-problems) is adopted here for specific $\{[\lambda] \rightarrow \Gamma(SU(m) \times \mathcal{S}_{20} \downarrow A_5)\ \}$ subductions, instead of full order algebraic formalisms, typical of $SU(m \leq 8) \times \mathcal{S}_{2n \leq 8}$ full multipartite problems, . The mappings derived are those developed over for a full for $SU(m = (2I_i + 1))$ set(s) of branchings (down to λ_{SA}). This work reports a sequel to that on branching rules for $SU(m \leq 7) \times \mathcal{S}_{12} \downarrow A_5$ embeddings of the $[^{10}B^1H]^{2-}_{12}$ spin system, set out in {Eurphys. J., **B11**, 177, (1999)}, cf. to { Physica **A227**, 314 (2002); J. Mol. Struct. (Theochem.), **578**, 145 (2002)} which stress the role of geometry (via invariance algebra) in spin physics.

KEYWORDS:

Uniform Multispin NMR Symmetry;

Branching rules for Direct Group Embeddings;

Lattice-point Set, Polyhedral Combinatorics in NMR;

Projective Modelling of Uniform higher I_i Spin Nano-structures.

1. Introduction

Algorithmic views of (multicolour) polyhedral combinatorics(PC)[1-6], applied to identical multispin NMR systems have special value in treating $\mathcal{S}_n$ -based NMR spin symmetry, since these are governed by $\hat{\mathbf{H}}$ (or their corresponding Liouvillians[7,8,]) scalar interactions, which themselves form interactive permutational networks[1]). This view (or the cycle-index approach of ref.[9]) allows some further useful insight into detailed nature of branching rules associated with direct subduction. In addition, it is these rules and their analytic determinacy as group embeddings that yield the $SU(m) \times \mathcal{S}_n \downarrow \mathcal{G}$ automorphic symmetries[1] characterising the identical multispin ensembles, discussed below. Various distinct aspects of these have been detailed elsewhere, i.e., in work dealing with either, $SU(2)$ branched Cayley group-order rule type systems (i.e. comparable to Akimova's views[2] applicable to (endohedral) $^{13}C_{60}$ fullerene NMR system[3]), or $m \sim n$ embedded forms whose branchings are amenible to use of linear algebra of §2 by virtue of their modest dimensionality, or finally to the $\lambda \vdash n$ hierarchical structuring of the strongly dominant $\mathcal{S}_{2n}$ form of direct embedding problem for some restricted specific $SU(m = 2I_i + 1)$ level - as discussed here in the context of $[^2HC]_{20}$, the comparable azadodecahedral spin system $[^{14}N]_{20}$ (postulated in ref. [10]), or the similar $[^{14}N_{20}C_{40}]$ azafullerene structure. For the latter method, the determinacy issue is taken as proven only IFF the mappings exhibit full 1:1 bijection that holds for all $\{[\lambda]\}$ subductional mappings, set down to λ_{SA}[11]. In all three cases, one draws on both the standard semi-normal techniques[12,13] for decomposition of Schur GL_n group irreps, as well as on the PC modelling of that entity in terms the quasi-geometric invariances, i.e., on the space of the automorphic group. The independence of these sets of invariances is central to the analytic determinacy question[11, 14-16], e.g. in comparing the $\mathcal{S}_8 \downarrow \mathcal{O}$, to $\mathcal{S}_{10} \downarrow \mathcal{D}_5$ or other similar spin symmetries. The $\mathcal{S}_{12} \downarrow A_5 \equiv \mathcal{I}$ embeddings, including all the $(\lambda) \trianglerighteq \{(\lambda)_{SA}\}$ of the $\{621^4;\ 53211;\ 4422\}$ self-associate (SA) (-conjugate) subset given in earlier work[11], clearly demonstrate that the branching rules for $[\lambda](\mathcal{S}_{12})$ subduction exhibit universally determinacy. This holds even for $SU(m \leq 7) \times \mathcal{S}_{12} \downarrow A_5$ symmetries related to $[^{10}B]_{12}$ NMR uniform spin subsystem-based embedding problem- in contrast to other specific low n,m- indexed systems[14].

On account of its large partitional branching -based dimensionality, the examination of $2n = 20$-indexed $SU(m) \times \mathcal{S}_{2n} \downarrow \mathcal{G}$ embeddings represents an especially interesting problem to treat via hierarchical sequence. The use of general multicolour PC decompositions and analogous projective methods is invoked in seeking mapping exhibiting analytic forms and universal determinacy. The non-matrix approach adopted is based on typical *inverse problem logic* applicable to the hierachical structure of Schur mappings, themselves based on $\lambda \vdash n$. Hence the $SU(m) \times \mathcal{S}_{20} \downarrow A_5 \equiv \mathcal{I}$ problem spans an extensive set, i.e., from $5^2 4^2 2$, down to and including the maximal 7-fold component of the $\{(\lambda_{SA})\}(\mathcal{S}_{20})$ subset (in hexadecimal), $A21^8$. For brevity, we shall confine our present report principally to the $SU(3) \times \mathcal{S}_{20} \downarrow A_5$ branching maps, corresponding to $[^2H]_{20}$, *or* $[^{14}N]_{20}$ NMR spin (sub)systems, with only the briefest mention made of the $SU(4)$ branched $I_i = 3/2$-related spin subsystems[1].

[1] Research into higher uniform spin nano-structures (met-cars) however could well render (say) $SU(6)$ (*of* $I_i = 5/2$) branching sets of interest in the future.

2. On $SU(m) \times \mathcal{S}_{2n} \downarrow \mathcal{G}$ Branching Rules in Contrasting Limits: a)all $m \leq$ (*modest*) $2n$ of the Matrix Formalism, vs b) the $2n \gg m \leq 3,4$ Models in a Hierachical Approach.

The first is concerned with modest indexed $|\mathcal{S}_{2n}|$ group order problems where all possible branched $m \leq 2n$ forms of (λ) decompositions may be treated readily via single-step matrix algebra; by contrast the b) (second) scenerio arises where the index is $2n \gg m$ and the group order based on the numbers of $\lambda \vdash$ index $2n$ for medium-to-high m values 'explodes', so that only a few m branched subsets are conveniently treated in terms of Schur decomposition process[12,13], using the non-matrix approach. The subductional mapping process:

$$[\lambda](\mathcal{S}_{2n}) \longrightarrow \Gamma(\mathcal{S}_{2n} \downarrow \mathcal{G}),$$

arises via (for the a) overall scenerio) a linear algebra which utilise inverses both of the Kostka matrix - itself based on completely inverse-dominance ordered $[\lambda]$s -, $\underline{K}^{-1}$, and of the $(\mathcal{G} \equiv A_5)$ character table[16] (written here as $\underline{T}(\underline{\chi}_{ij})$). Hence the branching rule sought in describing the $\Gamma(SU(m) \times \mathcal{S}_{2n} \downarrow \mathcal{G})$ direct subduction process becomes (in terms of standard vector/matrix notation):

$$(\ [\lambda]\ ) = \{\underline{K}^{-1}(\ \ (\lambda)\ \ )\}^{\dagger} \left\{\underline{T}^{-1}(\underline{\chi}_{ij})\,(\ \Gamma\ )\right\}; \tag{1}$$

in terms of the directly deduced invariances $\{\chi_i\}$ lying at the origin of the present calculations, this formal expression reduces to a simpler direct form:

$$(\ [\lambda]\ ) \equiv \{\underline{K}^{-1}(\ \ (\lambda)\ \ )\}^{\dagger}(\ \{\chi_i\}\ ); \tag{2}$$

Since this form highlights the origin of any indeterminacy in group embedding, as arising from degeneracy, or lack of independance, in the invariance sets over the $(C_i)(.. \downarrow \mathcal{G})$ algebra, this form has some conceptual advantage, both here and in discussions of §5. The subsequent work reported here is necessarily restricted to $m = 3(4)$ branchings, because the formal Kostka inverse matrix (i.e., for complete (λ) sets in reverse-dominance ordering on $\mathcal{S}_{20}$) is now of order $((1/2)(20!)) \sim 10^{18}$ so that direct matrix algebraic operations of method (a) are not invoked. Instead, it is convenient to utilise (b)-type modelling in investigating the form of the branching rule in subsequent calculations. One final comment of the algebraic approach is given here. This concerns the specific $\underline{T}(\underline{\chi}_{ij})$ for the group A_5 (i.e., from a classic induced group character discourse[17]) and its inverse. Although A_5 has no proper subgroup, it does have an interesting form of inverse, namely:

For A_5 group:

$$\underline{T}^{-1} = (1/60)\left\{\begin{array}{ccccc} 1 & 4 & 5 & 3 & 3 \\ 15 & 0 & 15 & -15 & -15 \\ 20 & 20 & -20 & 0 & 0 \\ 12 & -12 & 0 & -24/(1-\sqrt{5}) & -24/(1+\sqrt{5}) \\ 12 & -12 & 0 & -24/(1+\sqrt{5}) & -24/(1-\sqrt{5}) \end{array}\right\}. \tag{3}$$

On utilising the b) type modelling approach, it follows that once the full set of Kostka coefficients over enough (λ) Schur forms have been established by the methods of refs.[12,13], clearly one may invoke progressive hierachical difference set calculations over a series of specific pairs of (λ)s. The respective polyhedral combinatorial algebras follows directly, with the $\{\chi_i\}(\mathcal{S}_{2n} \downarrow \mathcal{G})$ invariance sets, as analogues of row-projections of Eq.(2) above. It is noted that this approach allows the lattice-point set-realised (λ) (PC) properties to be obtained by a number of analogous independant routes, so that the validity of the (inherent 1:1 bijective) mappings are obtained unambiguously via the $\{\chi_i\}(C_i(\mathcal{S}_{20} \downarrow A_5))$ invariance algebra(s) of Table 2; this reports Hilbert spin-alone -space results for the abstract automorphic spin symmetry. It is stressed that the abstract symmetry arises solely from inherent J_{ij} -interaction J hierarchies represented by the form of NMR Hamiltonian - there being no actual geometric space in some internal frame.

3. The 'sst'-tableaux Methods in (λ) Decompositions via $\{K_{\lambda\lambda'}\}$ sets.

Our initial discussion focuses on the descriptions of (λ) properties in terms of their polyhedral combinatorial -based invariance sets and also on their simple (or difference) Kostka decompositions; taken together these allow one to establish the form of $[\lambda]$ irrep mapping. Only partial tabulations, or illustrative examples, are given for reasons of brevity, with these being restricted to $SU(3 \leq m \leq 6) \times \mathcal{S}_{20} \downarrow A_5 \equiv \mathcal{I}$ branchings of explicit interest here. Details of $\mathcal{S}_n$- algorithmic techniques that yield the set of Kostka coefficients arising from insertion of any specific $p = part(s)$ $(\lambda)(\mathcal{S}_{20})$ of stated branching (and hence of known $\overbrace{1,1,1,..;2,2..;3,..}$, etc -index labelling) into the set of box tableaux corresponding to λ' members of the decompositional (irrep) set $[\lambda']$ for such index labelling which is constrained to a enumerated form corresponding to some *standard semi-normal* (sst) tableaux layout. This has been clearly described by Sagan[12], in his 1991 monograph; alternatively, the $\{K_{\lambda\lambda'}\}$ sets are obtainable directly from the *SYMMETRICA* (discrete mathematics) package, due to Kerber et al [13]. It should suffice here to present *illustrative* examples of the various Kostka coefficient $\{K_{\lambda\lambda'}\}$ sets on $\{[\lambda']\}$ associated with specific m-levels of $SU(m) \times \mathcal{S}_{20}$ branching, together with specific examples of their computational usage via multicolour (λ)-difference techniques in solving the inverse problem(s) associated with $SU(m = 3) \times \mathcal{S}_{20} \downarrow A_5 \equiv \mathcal{I}$ or else over similar $m = 4, 5, 6..$ branchings of analogous *identical* $I_i = 1,\ 3/2, ..(5/2)$ spins of 20-fold dual algebras.

In the following study of the weak branching of $(\lambda)(\mathcal{S}_{20})$ for $SU(3) \times \mathcal{S}_{20}$, the field $\mathcal{L}(\mathcal{S}_{20})$ holds the progressive sequence of $[\lambda']$ irreps over $p \leq 3$ parts with $(\lambda) \equiv (r_1 r_2 r_3)$ designated in the reduced notation by $(\lambda) \equiv \mathbf{r}_2\mathbf{r}_3$, with $r_1 = (n - \sum_{i \geq 2} r_i, ...,)$ leading portion suppressed; hence the requisite Schur decompositional mapping becomes:

$$(\lambda) \longrightarrow \{K_{\lambda,\lambda'}\}\mathcal{L}(\mathcal{S}_{20}), \tag{4}$$

with $\{K_{\lambda,\lambda'}\}$ being a unique set of Kostka coefficients which act on the column vector $\mathcal{L}$ symbol whose contents represent the ordered $\{[\lambda']\}$ (sub)set (from [0], $[\lambda]$ (down towards) $\trianglerighteq [\lambda_{SA}]$)- for convenience, only the next initial null $p = 4(5)$-based Kostka entries are shown in the numerical maps set out below. Thereafter, 'sst'- tableaux enumeration yields the following decompositional maps:

$$(\lambda) = (\mathbf{32}) \longrightarrow \{\ 1231; 320;\quad 2210;\quad 11100\ \}\mathcal{L}(\mathcal{S}_{20}), \tag{5}$$

$$(\mathbf{33}) \longrightarrow \{\ 1231;420;\quad 3310;\quad 2220;\quad 11101\ \}\mathcal{L}, \tag{6}$$

$$(\mathbf{43}) \longrightarrow \{\ 1231;420;\quad 4310;\quad 3320;\quad 22201;\quad 11101\ \}\mathcal{L}, \tag{7}$$

$$(\mathbf{62}) \longrightarrow \{\ 1231;320;\quad 3210;3210;\quad 3210;\quad 22100;\quad 1110000\ \}\mathcal{L}, \tag{8}$$

$$(\mathbf{53}) \longrightarrow \{\ 1231;420;\quad 4310;4320;\quad 33201;\quad 22201;\quad 1110100\ \}\mathcal{L}, \tag{9}$$

$$(\mathbf{44}) \longrightarrow \{\ 1231;420;\quad 5310;4420;\quad 33301;\quad 22202;\quad 1110101\ \}\mathcal{L}, \tag{10}$$

together with the intermediate $(\lambda) = (r_2 r_3)$-branched map forms for $r_1 + r_2 \sim n/2$ forms:

$$(\mathbf{63}) \longrightarrow \{\ 1231;420;\quad 4310;4320;\quad 43201;\quad 33201;\quad 2220100;\quad 1110100\ \}\mathcal{L}, \tag{11}$$

$$(\mathbf{54}) \longrightarrow \{\ 1231;420;\quad 5310;5420;\quad 44301;\quad 33302;\quad 2220201;\quad 1110101\ \}\mathcal{L}. \tag{12}$$

Beyond (**A**) these $SU(m=3)$-branched Schur maps represent more varied decompositions. Knowledge of these mapping properties down to (**76**) is required to complete the $SU(3) \times \mathcal{S}_{20} \downarrow A_5 \equiv \mathcal{I}$ algebra

$$(\mathbf{73}) \longrightarrow \{\ 1231;420;\quad 4310;4320;\quad 43201;43201;\quad 3320100;\quad 2220100;\quad 111010000\ \}\mathcal{L}, \tag{13}$$

$$(\mathbf{64}) \longrightarrow \{\ 1231;420;\quad 5310;5420;\quad 54301;44302;\quad 3330201;\quad 2220201;\quad 111010100\ \}\mathcal{L}, \tag{14}$$

$$(\mathbf{55}) \longrightarrow \{\ 1231;420;\quad 5310;6420;\quad 55301;44402;\quad 3330301;\quad 2220202;\quad 111010101\ \}\mathcal{L}, \tag{15}$$

together with (**92**) → (**83**) subsequence, based on initial $(\lambda) = (992)$ Schur and the (**84**) → (**66**) subset on further initial $(\lambda) = (884)$ fuller forms, together with the final (**76**) Schur form for $(\lambda) = (776)$, as a full description of (λ)s. These (λ) partitions yield the following additional decompositional maps within:

$$\left.\begin{array}{lcl} (\mathbf{92}) & & \{1231;320;3210;3210;\quad 321000;321000;32100000;\quad 3210000;121000000;(--)1000000\}\mathcal{L} \\ & \longrightarrow & \\ (\mathbf{83}) & & \{1231;420;4310;4320;\quad 432100;432010;43201000;\quad 3320100;122010000;(--)1010000\}\mathcal{L} \end{array}\right\}, \tag{16}$$

$$\left.\begin{array}{lcl} (\mathbf{74}) & & \{1231;420;5310;5420;\quad 543100;543020;44302010;\quad 3330201;122020100;(--)1010100\}\mathcal{L} \\ & \longrightarrow & \\ (\mathbf{65}) & & \{1231;420;5310;6420;\quad 653010;554020;44403010;\quad 3330302;122020201;(--)1010101\}\mathcal{L} \end{array}\right\}. \tag{17}$$

This tripartite (λ) decompositional mapping sequence concludes with the follow four mappings:

$$\left.\begin{array}{lcl} (\mathbf{84}) & & \{1231;420;5310;5420;\quad 543010;543020;5430201;\quad 3430201;123020100;(--)102010; \\ & & \qquad (----)1000\}\mathcal{L} \\ (\mathbf{75}) & & \{1231;420;5310;6420;\quad 653010;654020;5540301;\quad 3440302;123030201;(--)102021; \\ & & \qquad (----)1010\}\mathcal{L} \\ (\mathbf{66}) & \longrightarrow & \{1231;420;5310;6420;\quad 753010;664020;5550301;\quad 3440402;123030301;(--)102022; \\ & & \qquad (----)1011\}\mathcal{L} \\ (\mathbf{76}) & & \{1231;420;5310;6420;\quad 753010;764020;5650301;\quad 3450402;123040301;(--)102032; \\ & & \qquad (----)1021;(..)1\}\mathcal{L} \end{array}\right\}, \tag{18}$$

4. The $(\lambda)(SU(4) \times \mathcal{S}_{20})$ Mappings onto the $\{[\lambda]\}$ Set in terms of Kostka coefficients.

The further analogous $\{K_{\lambda\lambda'}\}$ Kostka sets for $SU(4) \times \mathcal{S}_{20}$, derived from $r_2 r_3 r_4$ quadapartite forms of $\lambda \vdash 20$ span the following decompositional sets:

$$\left.\begin{array}{lcl} (\mathbf{111}) & & \{1333; 121; \qquad\quad \}\mathcal{L} \\ (\mathbf{211}) & & \{1343; 341; \quad 12110\}\mathcal{L} \\ & \longrightarrow & \\ (\mathbf{311}) & & \{1343; 441; \quad 34110; \quad 12110000\}\mathcal{L} \\ (\mathbf{221}) & & \{1353; 561; \quad 35320; \quad 12211000\}\mathcal{L} \end{array}\right\}, \tag{19}$$

$$\left.\begin{array}{lcl} (\mathbf{411}) & & \{1343; 441; \quad 44110; \quad 341100; \quad 12110000\}\mathcal{L} \\ (\mathbf{321}) & \longrightarrow & \{1353; 661; \quad 57320; \quad 354210; \quad 12211100\}\mathcal{L} \\ (\mathbf{222}) & & \{1363; 781; \quad 69630; \quad 366330; \quad 12311201\}\mathcal{L} \end{array}\right\}, \tag{20}$$

together with the now $r_1(= n-7)$, $(r_2 r_3 r_4)$ -based subsequences:

$$\left.\begin{array}{lcl} (\mathbf{511}) & & \{1343; 441; \quad 44110; \quad 441100; \quad 34110000; \quad 12110000\}\mathcal{L} \\ & \longrightarrow & \\ (\mathbf{421}) & & \{1353; 661; \quad 67320; \quad 574210; \quad 35421100; \quad 12211100\}\mathcal{L} \end{array}\right\}, \tag{21}$$

and

$$\left.\begin{array}{lcl} (\mathbf{331}) & & \{1353; 761; \quad 79320; \quad 586310; \quad 35523200; \quad 122121010\}\mathcal{L} \\ & \longrightarrow & \\ (\mathbf{322}) & & \{1363; 881; \quad 8B630; \quad 6A9430; \quad 36733401; \quad 123122011\} \end{array}\right\} \tag{22}$$

where this final subsequence above now is given in *hexadecimal* (single-digit) notation, i.e., where $A \equiv 10, ..F \equiv 15$. A final $(\lambda : p = 4)$ -partite decompositional subset, for brevity, is invoked here which spans the $\{[\lambda']\}$ sets:

$$\left.\begin{array}{lcl} (\mathbf{521}) & & \{1353; 661; \quad 67320; \quad 674210; \quad 57421100; \quad 354211000; \quad 12211100000\}\mathcal{L} \\ (\mathbf{431}) & & \{1353; 761; \quad 7A320; \quad 7A6310; \quad 58733200; \quad 355242010; \quad 122121011000\}\mathcal{L} \\ & \longrightarrow & \\ (\mathbf{422}) & & \{1363; 881; \quad 8A630; \quad 8C9430; \quad 5AA43401; \quad 367344011; \quad 123122021100\}\mathcal{L} \\ (\mathbf{332}) & & \{1363; 981; \quad 9D630; \quad 8EB530; \quad 6B756601; \quad 367355032; \quad 12313201211\}\mathcal{L} \end{array}\right\} \tag{23}$$

By comparison with this final strongly branched (λ) Schur form, ending the (λ) *of* $p = 3$ $\{(\mathbf{84}) - (\mathbf{66})\}$ and $((7)76)$, the subsequent (λ) Schurs as $p = 4, (5)$ partite forms span $(\mathbf{644}) - (\mathbf{554})$, down to $(\mathbf{555})$ subset sequence. The first SA-form, $(\lambda)_{SA} = (54^2 2)$ arises at $p = 5$ partite branching level; the 6-part subsequence of (λ) decompositional maps is unusual, in being terminated by *two* forms: namely, $(662^4)_{SA}$, $(64^3 11)_{SA}$.

The model invariance properties associated with various the PC (λ) Schur forms are treated here in terms of polyhedral lattice-point sets, described for $(\lambda) : \ (p \leq 3)$ in Table 1. Such properties stress the role of algebraic geometry in NMR automorphic spin physics. Clearly by viewing the modelling in terms of lattice-point set for some specific axis choice and noting the maximal form of such equivalent sets for this particular C_i axis view, one may utilise a suitable 3 (4) multicolour labelling; thus one readily discerns the physical quasi-geometric principles inherent in this aspect of PC[2]. Clearly, even where the results obtained may be correlated subsequently

[2] other examples of similar multicolour techniques applied to PC may be found in the multinomial group representation / cycle-index work of Balasubramanian[6,9].

to some algebraic 'IFF' condition, the PC lattice-point set view of the invariance sets expresses the underlying principles based on the importance of (quasi-)geometric approaches in spin physics. On combining the $\{\chi_i\}(C_i)$ invariance algebras of Table 1, with a convenient SR (or compact) $\triangle(\lambda)$ decompositional difference mappings, based on comparing related types of Schur forms (as implied in the earlier discussion of method b) in §2), one may derive details of individual $[\lambda]\mathcal{S}_{2n}$ irrep subduction, as now mapped onto the $\mathcal{G} \equiv A_5$ automorphic group - without either the problematic high algebraic dimensionality question, or the need to obtain the complete set of Kostka coefficients for all branchings (including those above the $SU(m)$ form of interest) arising. The full $[\lambda]$ subduction mappings are presented in Table 2. This method of local projection of compacted information content is necessarily limited in its application.

Analogous hierarchical techniques to the above decompositional process readily establish the nature of the initial $SU(4) \times \mathcal{S}_{2n=20} \downarrow A_5$ irrep mappings, where (after Harter & Weekes[18]) the unit column labels are given as $(\Gamma') \equiv (\mathcal{A}, \mathcal{G}, \mathcal{H}, \mathcal{T}_1, \mathcal{T}_3)(..\mathcal{S}_{20} \downarrow A_5)$, so with $(\chi)^\dagger (= \{\chi_i\}(C_i)(A_5))$, now as a unit column vector, it follows that:

$$\begin{pmatrix} [111] \\ [211] \\ [311] \end{pmatrix} \equiv \left\{ \begin{array}{rrrrr} 969 & 9 & 6 & -1 & -1 \\ 11475 & -45 & 0 & 0 & 0 \\ 67184 & 80 & -1 & 4 & 4 \end{array} \right\} (\chi)^\dagger (C_i)$$

$$\rightarrow \begin{pmatrix} 20 & 67 & 81 & 46 & 46 \\ 180 & 765 & 945 & 585 & 585 \\ 1141 & 4477 & 5619 & 3340 & 3340 \end{pmatrix} \Gamma'^\dagger, \tag{24}$$

and also:

$$\begin{pmatrix} [221] \\ [411] \\ [321] \end{pmatrix} \equiv \left\{ \begin{array}{rrrrr} 56525 & -35 & 5 & 0 & 0 \\ 250800 & -240 & 15 & 0 & 0 \\ 408576 & 0 & -24 & -4 & -4 \end{array} \right\} (\chi)^\dagger,$$

$$\rightarrow \begin{pmatrix} 935 & 3770 & 4700 & 2835 & 2835 \\ 4125 & 16725 & 20835 & 12600 & 12600 \\ 6800 & 27232 & 34056 & 20428 & 20428 \end{pmatrix} \Gamma'^\dagger, \tag{25}$$

prior to our final illustrative mapping based on the [222] irrep:

$$[222] \equiv \{\ 129675 \quad 315 \quad 18 \quad 0 \quad 0\ \} (\chi)^\dagger$$

$$\rightarrow (\ 2246 \quad 8651 \quad 10879 \quad 6405 \quad 6405\ ) \Gamma'^\dagger, \tag{26}$$

6. Brief Concluding Remarks

The model projections and mappings given in Table 2 establish the complete $SU(3) \times \mathcal{S}_{20} \downarrow A_5$ subduction process associated with direct embedding to this specific level of branching and the fact that it is indeed analytically determinate (by virtue of the process representing sets of 1:1 bijective mappings). The Eqs.(24-26) above provide an illustration of the extension of such techniques to $SU(4) \times \mathcal{S}_{20} \downarrow A_5$ subduction pertinent to the role of identical I_i spins (and NMR) in high-spin nano-structures.

Schur functions related to NMR (i.e., beyond their use in describing completeness of identical higher I_i spin algebras via compositional catalogues, as given (e.g.) in ref [19]), the reader is referred to articles due to Sullivan & Siddall -III [20], and to the early work of Siddall-III[21] -on a comparable unitary view of identical higher $\{I_i\}$ multispin ensembles. [3]. Finally, questions of group measures, or invariant cardinality and determinacy, in NMR tensorial sets lie at the centre of any discussion of democratic recoupling[23-27] and the treatment of multi-invariant identical multispin systems, highlighting the inherent limits[28,29] to established (AMP) graphical recoupling theories[30].

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(λ) Red.Schur	$E = (.)$ *or* $(.)(.)$:	$C_{(12)(34)}$	$C_{(123)}$	$C_{(1-5)}$	$C_{(1-5)'}$
(0)	1	1	1	1	1
(1)	20	0	2	0	0
(2)	190	10	1		
(11)	380	0	2		
(3)	1140	0	6		
(21)	3420	0	0		
(4)	4845	45	12		
(31)	19380	0	12	0	0
(22)	29070	90	0	0	0
(5)	155040	0	6	4	4
(41)	77520	0	12		
(32)	155040	0	6		
(6)	38760	120	15		
(51)	232560	0	0		
(42)	581400	360	0		
(33)	775200	0	30		
(7)	77520	0	30		
(61)	542640	0	30		
(52)	1,627920	0	0		
(43)	2,713200	0	60		
(8)	125970	210	15	0	0
(71)	1,007760	0	30		
(62)	3,527160	840	15		
(53)	7,054320	0	30		
(44)	8,817900	1260	60	0	0
(9)	1,511640	0	20		
(81)	1,511640	0	0		
(72)	6,046560	0	0		
(63)	14,108640	0	60		
(54)	21,162960	0	0		
(A)	184756	252	40	6	6
((A)91)	1,847560	0	40		
(82)	8,314020	1260	0		
(73)	22,170720	0	120		
(64)	38,798760	2520	120		
(55)	46,558512	0	0	12	12
((9)92)	9,237800	0	20		
(83)	27,713400	0	60		
(74)	55,426800	0	120		
(65)	77,597520	0	60		
((8)84)	62,355150	3150	0		
(75)	99,768240	0	0		
(66)	116,396280	4200	90		
(76)	133,024320	0	180	0	0

Table 1: The reduced $(\lambda) \doteq (r_2 r_3)$ Schur forms in terms of Model Invariances on $(C_i)(\mathcal{S}_{20} \downarrow A_5)$ Algebra; the identity E is either a simple combinatorial, or else a monomial form. By contrast, the tabulated invariances on C_i derive from choices of a number of sub-sets from one of the axis-defined appropriate totals, i.e. that for total number of such sets of equivalent lattice points for relevant $C_i(A_5)$ axis-viewed dodecahedral 'Polyhedral Combinatorial'(PC) modelling.

$<\lambda>$ ([λ])	$\chi^{[\lambda]}_{1^n}$	$C_{(12)(34)}$	$C_{(123)}$	$C_{(-5)}$	$C_{(-5)'}$	:	A	G	H	T_1	T_3
$<0>$	1	1	1	1	1	:	1				
$<1>$	19	-1	1	-1	-1	:	0	2	1	1	1
$<2>$	170	10	-1	0	0	:	5	11	17	6	6
$<11>$	171	-9	0	1	1	:	1	11	12	11	11
$<3>$	950	-10	5	0	0	:	15	65	75	50	50
$<21>$	1920	0	-6	0	0	:	30	126	162	96	96
$<4>$	3705	45	6	0	0	:	75	249	318	174	174
$<31>$	11305	-35	1	0	0	:	180	754	933	574	574
$<22>$	7600	80	-5	0	0	:	145	505	655	360	360
(new)$\longrightarrow$											
$<5>$	10659	-45	-6	4	4	:	166	707	879	545	545
$<41>$	43776	0	0	-4	-4	:	728	2920	3648	2188	2188
$<32>$	55575	-45	0	0	0	:	915	3705	4620	2790	2790
$<6>$	23256	120	9	-4	-4	:	419	1555	1965	1132	1132
$<51>$	121125	-75	-15	0	0	:	1995	8070	10080	6075	6075
$<42>$	223839	315	0	4	4	:	3811	14921	18732	11114	11114
$<33>$	125970	-270	24	0	0	:	2040	8406	10422	6366	6366
$<7>$	38760	-120	15	0	0	:	621	2589	3195	1968	1968
$<61>$	248064	0	6	4	4	:	4138	16538	20670	12404	12404
$<52>$	604656	-240	-9	-4	-4	:	10013	40309	50331	30292	30292
$<43>$	620160	0	24	0	0	:	10344	41352	51672	31008	31008
$<8>$	48450	210	-15	0	0	:	855	3225	4095	2370	2370
$<71>$	377910	-90	0	0	0	:	6276	25194	31470	18918	18918
$<62>$	1,162800	720	0	0	0	:	19560	77520	97080	57960	57960
$<53>$	1,705440	-480	45	0	0	:	28319	113711	141985	85392	85392
$<44>$	872100	900	0	0	0	:	14760	58140	72900	43380	43380
$<9>$	41990	-210	5	0	0	:	649	2801	3445	2152	2152
$<81>$	413440	0	-20	0	0	:	6884	27556	34460	20672	20672
$<72>$	1,598850	-630	0	0	0	:	26490	106590	133080	80100	80100
$<63>$	3,100800	0	30	0	0	:	51690	206730	258390	155040	155040
$<54>$	2,848860	900	-60	0	0	:	47686	189904	237650	142218	142218
[AA]	16796	252	20	6	6	:	352	1124	1456	778	778
$<91>$	277134	-42	15	-6	-6	:	4611	18483	23079	13866	13866
$<82>$	1,469650	1050	-5	0	0	:	24755	97975	122735	73220	73220
$<73>$	3,779100	-420	45	0	0	:	62895	251955	314805	189060	189060
$<64>$	5,038800	1680	15	0	0	:	84405	335925	420315	251520	251520
$<55>$	2469012	-2580	-75	12	12	:	40485	164571	205131	124098	124098
[992]	604656	-1008	0	6	6	:	9828	40308	50136	30486	30486
$<83>$	2,687360	0	-40	0	0	:	44776	179144	223960	134368	134368
$<74>$	5,290740	-2580	-45	0	0	:	87519	352701	440265	265182	265182
$<65>$	4,837248	0	-120	-12	-12	:	80576	322448	403144	241860	241860
[884]	2,309450	1890	-40	0	0	:	38950	153950	192940	115000	115000
$<75>$	4,157010	690	45	0	0	:	69471	277149	346575	207678	207678
$<66>$	2,217072	1680	150	12	12	:	37426	147850	185126	110435	110436
[776]	1,385670	-1050	30	0	0	:	22842	92388	115200	69546	69546

Table 2: The Group Branching expressed as $[\lambda](SU(m \leq 3) \times \mathcal{S}_{20}) \longrightarrow \Gamma(\mathcal{S}_{20} \downarrow A_5)$ (bijective) mappings (i.e., on the basis of the independent non-degenerate forms of model invariances sets). The first 9 entries were given in an earlier work[8] on $[\lambda](SU(2) \times \mathcal{S}_{20})$ embedding. The respective Butler-Wybourne and Harter & Weekes[24] notations for reduced $[\lambda]\mathcal{S}_n$, and A_5 irreps are retained both here and in the main text.